Emoji as a Formal View: A Two-Pass Pipeline for Translating Abstract Algebra into a Pure-Iconic Language

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This paper proposes and evaluates a two-pass "Rosetta→Emoji" pipeline that translates standard abstract algebra into a pure-iconic language while preserving meaning and proof flow. Pass 1 (Rosetta) presents copy-safe math and English—definitions, lemmas, micro-diagrams, and fixed proof moves (Construct, Compose, Invert, Fill, Witness, Quotient). Pass 2 (Emoji) deterministically encodes every token, arrow, and diagram tag into an emoji alphabet governed by a simple grammar (precedence, typing hints, and universal-property tags). Using examples from this project—First Isomorphism Theorem (groups), Correspondence and CRT (rings), orbit–stabilizer and semidirect products, finite-field facts, Smith normal form, and basic categorical limits—we demonstrate line-for-line mirrors and round-trip decoding. We then analyze where full conversion fails: variable binding and quantifiers, 2D diagram semantics, universe accounting, and the equality/equivalence split. The result is a practical method for teaching and reviewing algebra with screenshot-ready pages, paired with a sober assessment of foundational limits. We close with a roadmap for future systems: a verified formal core, parametric tokens with real binding, a diagram engine, equivalence-first logic, and bundled icon fonts to make the emoji view robust and platform-independent.

Keywords: abstract algebra, emoji language, formal methods, groups, rings, ideals, homomorphisms, kernel, image, quotient, isomorphism theorems, group actions, semidirect product, Chinese Remainder Theorem, fields, extensions, finite fields, modules, Smith normal form, categories, products, equalizers, pullbacks, universal properties, grammar, encoding, pedagogy.

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1. Introduction and Contributions

1.1 Motivation

Mathematical exposition balances precision and readability. Traditional notation is precise but can be dense for newcomers; purely visual interfaces are approachable but often ambiguous. This paper explores a *two-pass pipeline* that renders standard abstract algebra (Rosetta: copy-safe math + English) into a *pure-iconic* form (Emoji: a constrained emoji alphabet with a grammar). The aim is didactic: students learn from Rosetta, then reread the same content in emoji to practice structure recognition without relying on familiar letter-strings.

1.2 Problem Statement

We ask: **How far can core abstract algebra be translated into a lossless, platform-portable emoji language that preserves proof flow and universal constructions?** Specifically, we target:

- groups $(G,*)$, homomorphisms, kernels/images, isomorphism theorems;
- rings $(R,+,\cdot)$, ideals, quotients, CRT;
- fields/finite fields, simple extensions, basic Galois sketches;
- modules and Smith normal form over a PID;
- categorical limits/colimits and universal properties.

1.3 The Two-Pass Method (Rosetta→Emoji)

- **Pass 1: Rosetta.** Write statements and proofs with copy-safe ASCII math and minimal diagrams, using a *six-move library*: Construct (C), Compose (K), Invert (I), Fill (F), Witness (W), Quotient (Q). Each proof line is atomic: *[Move] statement — (justification)*.
- **Pass 2: Emoji.** Deterministically encode every token (objects, operations, logic, diagram tags) into an emoji alphabet. An unambiguous grammar enforces precedence, typing hints, and attachment of universal-property tags. Line numbers and step structure are preserved, enabling round-trip (decode(encode(x)) $=$x).

1.4 Contributions

1. **Pipeline Design.** A practical, deterministic Rosetta→Emoji translator for a broad undergraduate algebra curriculum, including commuting-square “micro-diagrams” and universal-property annotations.
2. **Encoding Table & Grammar.** A compact token set and grammar covering objects (groups, rings, fields, modules, categories), operations (product, semidirect, direct sum),

logic (equality, implication, iff, existence/uniqueness), and diagram constructors (product, equalizer, pullback/pushout).

3. **Move Templates.** Reusable proof macros (C/K/I/F/W/Q) that compress standard arguments: First Iso, Correspondence, CRT(2), product UP, orbit–stabilizer, SNF over a PID.
4. **Line-for-Line Case Studies.** Fully mirrored pages (Rosetta vs Emoji) for: First Iso (groups), Correspondence + First Iso (rings), CRT with explicit idempotents, Cayley + semidirect product, finite-field facts, SNF (PID), product/equalizer UPs.
5. **Validation Protocol.** A lightweight parsing/typing procedure that lexes emoji, checks constructor types and tags (e.g., normality before quotient), audits diagrams, and verifies round-trip.
6. **Limit Analysis.** A precise account of where full conversion fails: variable binding and quantifiers, 2D diagram semantics, universe accounting, and the equality/equivalence split central to category theory and HoTT.
7. **Roadmap.** Requirements for a future system that *completes* the vision: verified formal core, parametric glyphs with true binding, a 2D diagram engine, equivalence-first logic, and bundled icon fonts for cross-platform stability.

1.5 Scope and Non-Goals

- **In scope.** Undergraduate to early graduate algebra proofs with short diagram chases; universal properties used as definitions; basic field theory and module theory; compact categorical limits/colimits.
- **Out of scope (for now).** Full higher-categorical treatments, HoTT/∞-groupoids with coherence towers, large/stacked commuting diagrams that require native 2D semantics, and meta-theory involving universe hierarchies.

1.6 Representative Examples (from our working pages)

- **First Isomorphism (Groups).** Rosetta: $f: G \to H$homomorphism $\Rightarrow G/\ker\ f \cong \mathrm{im}\ f$. Emoji mirror uses tokens for kernel ⚫, image 🌈, quotient 🖇️, and an 8-line W/Q/K sequence.
- **CRT (Two Ideals).** Rosetta: $I + J = R \Rightarrow R/(I \cap J) \cong R/I \times R/J$; with explicit e_I, e_J. Emoji mirror marks comaximality (🤝), quotient, product, and supplies the inverse via $ue_I + ve_J$.

- **Cayley & Semidirect.** Rosetta: left-regular action $G \hookrightarrow \mathrm{Sym}(G)$; $N \rtimes_{\varphi} K$with $(n,k)(n',k') = (n\,\varphi(k)(n'), kk')$. Emoji mirror encodes actions, injections, normality, and the φ-controlled product.
- **SNF (PID).** Rosetta: $PAQ = \mathrm{diag}(d_i)$with divisibility chain. Emoji mirror compresses the row/column operation loop and the invariant factor decomposition.

1.7 Methodological Principles

- **Determinism first.** One Rosetta string → one emoji string; grammar chooses longest-match tokens; universal tags (🏆 , mono/epi tags, action controllers) attach to specific constructors.
- **Atomic steps.** Each proof line claims one fact with one justification. This supports didactic clarity and machine-checkable parsing.
- **UP-centric definitions.** Products, equalizers, pullbacks, etc., are defined by their universal properties; emoji tags encode uniqueness and cone/coecone roles.
- **Minimal alphabet.** Tokens are deliberately few; new structures require table entries and tests (Appendix A).

1.8 What This Paper Adds to the Literature

Unlike ad-hoc iconography or informal “diagrammatic” proofs, our approach enforces a grammar and validates typing/UP constraints. Unlike proof assistants, our emoji layer is intentionally *didactic*, not a foundation. The novelty is the **middle ground**: a human-readable, screenshot-friendly view backed by enough structure to be parsable and round-trippable for a large swath of algebra.

1.9 Document Organization

Section 2 fixes minimal notation and the six proof moves. Section 3 presents the encoding table; Section 4 the grammar and validator. Section 5–7 cover groups and actions through Sylow. Sections 8–11 address rings, fields, modules/SNF, and categorical limits. Section 12 collects naturality and UP patterns. Section 13 shows mirrored pages. Section 14 reports validation and edge cases. Section 15–17 give authoring guidelines, assessment, and extensibility. We conclude with limits and a forward path toward a formal core and a robust iconic front-end.

2. Background: Algebraic Structures and Minimal Notation

This section fixes copy-safe notation for the structures we use throughout and introduces the six proof moves that drive our Rosetta→Emoji translation. All symbols are ASCII-safe.

2.1 Sets, Functions, and Diagrams (micro)

- A function is $f: X \to Y$. Composition is $g \circ f$ with types $X \xrightarrow{f} Y \xrightarrow{g} Z$. Identity on X is id_X.
- A commutative square is the assertion that two composites agree:

$$X \xrightarrow{f} Y \xrightarrow{u} Z, X \xrightarrow{g} Y' \xrightarrow{v} Z, \text{and } u \circ f = v \circ g.$$

- Product of sets: $X \times Y$ with projections $\pi_X: X \times Y \to X$, $\pi_Y: X \times Y \to Y$. Universal property (UP): for any $h: Z \to X$ and $k: Z \to Y$ there is a unique $\langle h, k \rangle: Z \to X \times Y$ with $\pi_X \circ \langle h, k \rangle = h$ and $\pi_Y \circ \langle h, k \rangle = k$.
- Equalizer of $f, g: X \to Y$ is $e: E \to X$ where $f \circ e = g \circ e$ and it is universal among such arrows.

Conventions: we prefer stating objects via their UPs (product, equalizer, pullback/pushout) rather than elementwise definitions, since the Emoji layer attaches an explicit UP tag to the head constructor.

2.2 Groups and Actions

- Group: $(G, *)$ with identity e, inverses x^{-1}, associativity $(xy)z = x(yz)$.
- Subgroup: $H \leq G$. Normal subgroup: $N \lhd G$ iff $gNg^{-1} = N$ for all $g \in G$.
- Homomorphism: $f: G \to H$ with $f(xy) = f(x)f(y)$. Kernel $\ker f = \{x \in G: f(x) = e\}$. Image $\mathrm{im}\, f = \{f(x): x \in G\}$.
- Cosets and index: for $H \leq G$ and $a \in G$, $aH = \{ah: h \in H\}$. The index $[G: H]$ is the number of left cosets when finite.
- Quotient group: if $N \lhd G$, then G/N with $[a] \cdot [b] = [ab]$.
- First Isomorphism Theorem (groups): for $f: G \to H$, $G/\ker f \cong \mathrm{im}\, f$.
- Actions: a (left) action $G \curvearrowright X$ is a homomorphism $G \to \mathrm{Sym}(X)$. Orbit $\mathrm{Orb}(x) = \{g \cdot x\}$. Stabilizer $\mathrm{Stab}(x) = \{g: g \cdot x = x\}$. Orbit–stabilizer (finite): $|\ \mathrm{Orb}(x)\ | = [G: \mathrm{Stab}(x)]$.

- Direct product: $G \times H$componentwise. Semidirect product $N \rtimes_{\varphi} K$from an action $\varphi: K \to \mathrm{Aut}(N)$with $(n,k)(n',k') = (n\,\varphi(k)(n'), kk')$.
- Sylow (statement only): If $|\ G\ | = p^a m$with $p \nmid m$, there exist subgroups of order p^a; any two are conjugate; their number $n_p$divides $m$and $n_p \equiv 1 (\mathrm{mod} p)$.

2.3 Rings, Ideals, CRT

- Ring: $(R, +, \cdot)$with $(R, +)$abelian, $\cdot$associative, distributivity of $\cdot$over $+$. We use unital rings when needed; then $1_R$is preserved by unital homomorphisms.
- Ideal: $I \lhd R$means additive subgroup with $rI \subseteq I$and $Ir \subseteq I$for all $r \in R$. Principal ideal: $(a) = \{ra : r \in R\}$.
- Quotient ring: $R/I$with $[a] + [b] = [a + b]$, $[a][b] = [ab]$.
- Correspondence Theorem (rings): ideals of Rcontaining Icorrespond to ideals of R/I.
- First Isomorphism (rings): for a ring homomorphism $\phi: R \to S$, $R/\ker\ \phi \cong \mathrm{im}\ \phi$.
- Chinese Remainder Theorem (two ideals): if $I + J = R$, then $R/(I \cap J) \cong R/I \times R/J$. Constructive CRT: there exist $e_I, e_J \in R$with $e_I \equiv 1 (\mathrm{mod} I)$, $e_I \equiv 0 (\mathrm{mod} J)$, and $e_I + e_J = 1$, giving explicit inverse maps.

2.4 Fields, Extensions, Finite Fields, Galois (sketch)

- Field extension: $F \subseteq E$. Degree $[E : F]$is the dimension of Eas an F-vector space.
- Simple extension: $E = F(\alpha) \cong F[x]/(m_\alpha)$, where $m_\alpha$is the minimal polynomial.
- Tower law: $[E : F] = [E : K][K : F]$.
- Finite fields: for each prime power $q = p^n$there exists a field $\mathbb{F}_q$unique up to isomorphism, with $\mathbb{F}_q^{\times}$cyclic of order $q - 1$.
- Splitting field: the smallest extension in which a polynomial factors into linear terms; exists and is unique up to isomorphism.
- Galois sketch: for finite normal, separable E/F, the automorphism group $\mathrm{Gal}(E/F)$corresponds anti-isomorphically to the lattice of intermediate fields via fixed fields E^H.

2.5 Modules and Smith Normal Form (PID)

- Left R-module: an abelian group $(M, +)$with scalar action $R \times M \to M$satisfying distributivity, associativity, and $1 \cdot m = m$in the unital case.
- Module homomorphism $f: M \to N$preserves addition and scalar multiplication. A sequence $L \xrightarrow{f} M \xrightarrow{g} N$is exact at Mif $\ker\ g = \mathrm{im}\, f$.
- Free module R^n: admits a basis of $n$elements. Presentations are given by maps $R^m \xrightarrow{A} R^n$.
- Over a PID, every matrix admits a Smith Normal Form: there exist invertible $P, Q$with $PAQ = \mathrm{diag}(d_1, \ldots, d_r, 0, \ldots, 0)$and $d_i \mid d_{i+1}$. Structure theorem: $M \cong R^r \oplus \oplus_i R/(d_i)$.

2.6 Proof Moves, Step Format, and Conventions

We write each proof as a sequence of *atomic* lines, each in the template

[Move] statement — (justification)

with a single justification per line. The six moves are:

- **C = Construct.** Introduce an object, map, or diagram with minimal data.
  Example: define $\varphi([g]) = f(g)$on $G/\ker\ f$.
- **K = Compose.** Chain known maps or operations; record domain/codomain once.
  Example: use $\pi_X \circ \langle f, g \rangle = f$.
- **I = Invert.** Apply inverse or cancellation where defined (group inverse, unit, isomorphism inverse).
  Example: from $ax = ay$deduce $x = y$by left-multiplying a^{-1}.
- **F = Fill.** Invoke a universal property to obtain the unique mediating arrow.
  Example: produce $\langle f, g \rangle$to a product.
- **W = Witness.** Assert one atomic fact with a short reason (definition, prior lemma).
  Example: "$\ker\ \pi = I$" — (definition of quotient map).
- **Q = Quotient.** Pass to cosets/ideals/coequalizers and prove well-definedness.
  Example: check $[ab]$is independent of representatives in R/I.

Line economy. We aim for 8–12 lines per lemma/theorem. If a universal property or exactness condition is used repeatedly, label it once and reuse the label. Diagrams are kept to squares/triangles with explicit boundary objects listed before arrow equalities.

Minimal alphabet. We restrict to: groups $(G,*)$, rings $(R,+,\cdot)$, fields $F \subseteq E$, modules R-Mod, category objects/arrows with $\mathrm{id}$and $\circ$. All additional structure (e.g., semidirect products, equalizers, pullbacks) is introduced only when needed and referenced by its UP.

3. Method: Two-Pass Rosetta→Emoji Pipeline

This section specifies the pipeline end-to-end: tokenization of Rosetta pages, deterministic encoding to emoji, well-formedness via a grammar, and round-trip validation. We use only copy-safe math and ASCII punctuation.

3.1 Overview of Data Flow

Inputs. A Rosetta page: numbered section headers, atomic proof lines in the template [Move] statement — (justification); micro-diagrams (squares/triangles) with boundary lists.

Outputs. An Emoji page with identical numbering and line boundaries, where each Rosetta token is replaced by a unique emoji token, and each diagram is linearized with tags.

Invariant. For every Rosetta page R, decode(encode(R)) = R provided Ruses only tokens registered in the encoding table (Sec. 3.3) and conforms to the move grammar (Sec. 3.4).

3.2 Encoder/Decoder Contracts

We model the pipeline as partial functions:

- encode : Rosetta → Emoji (partial, total on “well-formed Rosetta”)
- decode : Emoji → Rosetta (partial, total on images of encode)

Determinism. If encode is defined on R, the result is unique. If decode is defined on E, the result is unique.

Round-trip. For well-formed R: decode(encode(R)) = R.
No spurious fixed points. For any $E$with decode(E) defined, encode(decode(E)) = E.

3.3 Encoding Table (Core Map)

A *token map* τassigns each Rosetta token a unique emoji token:

- Objects and roles:
  $G \mapsto$ 🧊, $R \mapsto$ 🧲, $F \mapsto$ 🟦, $E \mapsto$ 🟩, $M \mapsto$ 📦
  element symbols $g, h, x, y \mapsto$colored dots 🔴 🔵 🟡 🟣 (as needed)
  identity $e \mapsto$ ⭐
- Operations and relations:
  group op $* \mapsto$ ⛓️, inverse $(\cdot)^{-1} \mapsto$ 🔁, product $\times \mapsto$ ✖️, semidirect $\rtimes \mapsto$ 🪝, direct sum $\oplus$ $\mapsto$ ➕ ⭕, big sum $\bigoplus \mapsto$ ➕ 🟠, intersection $\cap \mapsto$ 🌀, inclusion $\subseteq \mapsto$ 🔻, normality $\triangleleft \mapsto$ 🛡️
- Maps and structure:
  homomorphism symbol $f, \phi \mapsto$ 🪄, kernel $\ker \mapsto$ ⚫, image $\mathrm{im} \mapsto$ 🌈, projection $\pi \mapsto$ 🕹️, inclusion $\iota \mapsto$ 🕹️, quotient bar "/" $\mapsto$ 🪐, quotient map tag $\pi: G \to G/N \mapsto$ 🗺️
- Logic and flow:
  equality $= \mapsto$ 🟰, implication $\Rightarrow \mapsto$ ➡️, iff $\Leftrightarrow \mapsto$ 🔁, existence $\exists \mapsto$ ✨, uniqueness $! \mapsto$ 🔒, contradiction $\bot \mapsto$ 💥
- Universal constructions (with UP tag 🏆):
  product, equalizer, coequalizer, pullback, pushout $\mapsto$ ✖️🏆, ➖🏆, ➕🏆, ⬇️🏆, ⬆️🏆

Multi-glyph atoms. Some tokens are *atomic sequences*, e.g., ➕🟠, not two independent glyphs. The lexer uses longest-match (Sec. 3.5).

Extensibility. New structures (e.g., lattices) require adding tokens and tests (Appendix A).

3.4 Move Library (Verb Set)

Each proof line begins with exactly one move tag:

- **C (Construct).** Introduce object/map/diagram with minimal data.
- **K (Compose).** Compose arrows or operations with declared domains/codomains.
- **I (Invert).** Use inverses/units/isomorphism inverses or cancellation.
- **F (Fill).** Invoke a universal property to obtain the mediating arrow.
- **W (Witness).** Assert a single atomic fact with a short justification.

- **Q (Quotient).** Pass to a quotient and prove well-definedness.

Normalization. If a line contains several actions, split into multiple lines or refactor through W/K/F as needed. This enforces atomicity and aids decoding.

3.5 Grammar (Well-Formed Emoji)

We specify a BNF-style grammar; terminals are emoji tokens.

Line ::= "[" Move "]" " " Statement " — " Just

Move ::= C | K | I | F | W | Q ; rendered as 🧩 ⛓️ 🔁 ⬜ 🪄 🔗

Statement ::= Term

| Term 🟰 Term

| Term ➡️ Term

| Term 🔂 Term

Term ::= Atom

| 🔁 Term

| Term ⛓️ Term

| Constr "(" TermList ")"

| Tag "(" Term ")"

Atom ::= 🧊 | 🧲 | 🟦 | 🟩 | 📦 | 🔴 | 🔵 | 🟡 | 🟣 | ⭐ | 🪄 | ⚫ | 🌈 | 📥 | 📥

Constr ::= ❌ | 🪝 | (🧊 🔗 🧺) | (🧲 🔗 🧺) | ➕ ⭕ | ➕ 🟠 | ➖ | ⬇️ | ⬆️

Tag ::= 🏆 | (mono) | (epi) | (left/right action) | (index/cardinality) ...

TermList ::= Term | Term "," TermList

Just ::= one of { "def", "UP", "ker", "im", "bij", "inj", "surj",

"well-def", "assoc", "comm", "nat", "PID", ... }

Precedence. Prefix 🔁 binds tightest; then constructors; then ⛓️ (left-assoc); finally relational operators (🟰, ➡️, 🔂). Tags are transparent for equality and composition.

Diagram linearization. A square lists boundary objects first, then arrow equalities, e.g. ☐(X,Y,Y',Z ; u∘f = v∘g) which encodes to a fixed emoji template with the same order.

3.6 Lexing and Longest-Match

Rule. When scanning emoji, prefer the longest valid multi-glyph token (e.g., ➕🟠 before ➕). This avoids ambiguity between big direct sum and direct sum.

Numbers and indices. We use emoji digits 0–9 and ASCII subscripts/superscripts only in Rosetta. In Emoji pages, cardinalities and indices are recorded as tagged metadata (e.g., #X, [G:H]) attached to lines; the decoder re-inserts them.

3.7 Typing and Sanity Checks

Before emission, encode performs lightweight typing checks:

- **Hom typing.** If a line contains ✨: 🧊➡️🧊 then ⚫, 🌈 must live in 🧊.
- **Quotients.** 🧊🖇️🧺 requires a normal/ideal tag introduced earlier on the same line or a prior line in scope.
- **Semidirect.** 🪝 requires an explicit controller tag (e.g., 👑 on the right factor).
- **UP scope.** A 🏆 tag attaches to exactly one head constructor per line.

Failure modes. If any check fails, encode halts with a diagnostic (Sec. 3.11).

3.8 Algorithms (Pseudocode)

3.8.1 Encode (Rosetta → Emoji)

```
function ENCODE(page R):
  assert is_atomic_lines(R)
  tokens = []
  for each line L in R in order:
    (move, stmt, just) = parse_line(L)        # uses Rosetta micro-grammar
    t_move  = tau(move)
```

```
  t_stmt  = encode_stmt(stmt)              # recursive on terms & diagrams
  t_just  = encode_just(just)              # canonical code
  check_typing(t_stmt)                     # Sec. 3.7
  tokens.push(format("[{t_move}] {t_stmt} — {t_just}"))
 return join(tokens, "\n")
```

3.8.2 Decode (Emoji → Rosetta)

```
function DECODE(page E):
 lines = lex_emoji(E, longest_match=True)      # Sec. 3.6
 R = []
 for each emoji line ℓ:
  (t_move, t_stmt, t_just) = parse_emoji_line(ℓ)
  r_move = tau^{-1}(t_move)
  r_stmt = decode_stmt(t_stmt)              # inverse on constructors/tags
  r_just = decode_just(t_just)
  R.push(format("[{r_move}] {r_stmt} — ({r_just})"))
 return normalize(R)                       # restore minimal Rosetta
```

3.8.3 Round-Trip Test

```
for each sample page S in corpus:
 assert DECODE(ENCODE(S)) == S
 assert ENCODE(DECODE(ENCODE(S))) == ENCODE(S)
```

3.9 Worked Micro-Example (First Iso, group version)

Rosetta (2 lines excerpt):

[W] ϕ([g]) = f(g) — (definition)

[Q] [g]=[h] ⇒ f(g)=f(h) — (well-def)

Emoji:

[🪄] ϕ([🔴]) 🟰 🪄 (🔴) — def

[🧬] [🔴]=[🔵] ➡️ 🪄 (🔴)= 🪄 (🔵) — well-def

Type check passes because ⚫ (f) was declared and 🧊 🧬 ⚫ is in scope.

3.10 Complexity and Limits

- **Time.** Linear in page length for lex/parse; typing checks are linear in term size; diagram audit is linear in the number of edges.
- **Space.** Linear in token count.
- **Expressive boundary.** The grammar handles first-order algebraic arguments and small diagram chases. It does not natively encode variable binding (∀, ∃ over families), large coherence diagrams, or universe bookkeeping—see Sec. 6 for failure analysis.

3.11 Diagnostics and Error Messages (Copy-Safe)

- Type mismatch at line k. → " 🍼 type error @ line k: domain/codomain mismatch."
- Missing well-definedness on quotient. → " 🧬 requires 'well-def' proof on the same or previous line."
- Naturality square not justified. → " ⬜ needs 'comm' or a reference to a prior commutative square."
- Controller missing for semidirect. → " 🪝 requires explicit action tag (right factor)."
- Ambiguous tokenization. → "Resolve clash between ➕ and ➕ 🟠 by explicit big-sum tag."

3.12 Tooling and Authoring Recommendations

- **Authoring.** Write proofs with the six moves; keep each line atomic and ≤80 characters. Label constructions once; reuse labels.
- **Corpus.** Maintain a small suite of mirrored pages (First Iso, Correspondence, CRT-2, Product UP, Orbit–Stabilizer, SNF(PID)) as regression tests.

- **Extending tokens.** Additions must include: token entry, grammar production (if new constructor), minimal typing rule, and two round-trip tests.

4. Case Studies (line-for-line mirrors)

We present representative theorems as *paired pages*: a Rosetta proof (copy-safe math + English, with move tags) and a deterministic Emoji mirror. Each proof is kept to atomic lines (≤80 chars) and uses the six moves C/K/I/F/W/Q.

4.1 First Isomorphism Theorem (Groups)

4.1.1 Rosetta page

Statement. If $f: G \to H$is a homomorphism, then $G/\ker\ f \cong \operatorname{im} f$.

[W] Define ϕ:G/ker f → im f by ϕ([g]) = f(g). — (def)

[Q] If [g]=[h], then g(ker f)=h(ker f) ⇒ f(g)=f(h). — (well-def)

[K] ϕ([gh]) = f(gh) = f(g)f(h) = ϕ([g])ϕ([h]). — (hom)

[W] im ϕ = im f. — (by def of ϕ)

[W] ker ϕ = { [g] : f(g)=e } = { [e] }. — (ker criterion)

[W] Hence ϕ is injective. — (ker ϕ trivial)

[F] Image ϕ = im f, so ϕ is surjective onto im f. — (def)

[W] Therefore G/ker f ≅ im f via ϕ. — (bij)

4.1.2 Emoji mirror

[🪄] ϕ:🧊 🪐 ⚫ ➡️ 🌈 ; ϕ([🔴])= 🪄(🔴) — def

[🪐] [🔴]=[🔵] ➡️ 🪄(🔴)= 🪄(🔵) — well-def

[⛓️] ϕ([🔴 ⛓️ 🔵])= 🪄(🔴 ⛓️ 🔵)= 🪄(🔴) ⛓️ 🪄(🔵)=ϕ([🔴]) ⛓️ ϕ([🔵]) — hom

[🪄] 🌈(ϕ)= 🌈(🪄) — def

[🪄] ⚫(ϕ)={ [🔴] | 🪄(🔴)= ⭐ }={ [⭐] } — ker

[🪄] ⇒ ϕ inj — ker= ⭐

[⬜] 🌈(ϕ)=🌈(🪄) ⇒ ϕ surj→🌈 — def

[🪄] 🧊🧬⚫🔁🌈 via ϕ — bij

4.2 Correspondence + First Iso (Rings)

4.2.1 Rosetta page

Statement A (Correspondence). Ideals J of R with $I \subseteq J$ correspond to ideals of R/I.

[C] π:R→R/I the projection; ker π = I. — (def)

[W] For J⊇I, set J' = π(J) ⊂ R/I; J' is an ideal. — (ideal image)

[W] For J'⊂R/I, set J = π^{-1}(J'); J is an ideal with I⊂J. — (preimage)

[W] These maps are inverse: π(π^{-1}(J'))=J', π^{-1}(π(J))=J. — (check)

[W] Thus ideals over I ↔ ideals of R/I. — (corr)

Statement B (First Iso). If $\phi: R \to S$ is a ring homomorphism, then $R/\ker\ \phi \cong \operatorname{im} \phi$.

[C] Define ϕ̄:R/ker ϕ → im ϕ by ϕ̄([a])=ϕ(a). — (def)

[Q] [a]=[b] ⇒ a−b∈ker ϕ ⇒ ϕ(a)=ϕ(b). — (well-def)

[K] ϕ̄([ab])=ϕ(ab)=ϕ(a)ϕ(b)=ϕ̄([a])ϕ̄([b]). — (hom)

[W] ker ϕ̄ = { [a] : ϕ(a)=0 } = { [0] }. — (ker)

[W] im ϕ̄ = im ϕ. — (def)

[W] Hence ϕ̄ bijective; R/ker ϕ ≅ im ϕ. — (bij)

4.2.2 Emoji mirror

[🧩] 🗺️:🧲➡️🧲🧬🧺 ; ⚫=🧺 — def

[🪄] J⊇🧺 ⇒ J' = 🗺️(J) ⊂ 🧲🧬🧺 — ideal

[🪄] J'⊂🧲🧬🧺 ⇒ J = 🗺️^{-1}(J') ; I⊂J — preimage

[🪄] 🗺️(🗺️^{-1}(J'))=J' , 🗺️^{-1}(🗺️(J))=J — inv

[🪄] ideals over 🧺 🔁 ideals of 🧲🧬🧺 — corr

[🧩] ϕ:🧲🖇⚫➡️🌈 ; ϕ([🟣])=🪄(🟣) — def

[🖇] [🟣]=[🟡] ⇒ 🪄(🟣)=🪄(🟡) — well-def

[⛓] ϕ([🟣❌🟡])=…=ϕ([🟣])❌ϕ([🟡]) — hom

[🪄] ⚫(ϕ)={ [🟣] | 🪄(🟣)=0️⃣ }={ [0️⃣] } — ker

[🪄] 🌈(ϕ)=🌈(🪄) — def

[🪄] 🧲🖇⚫🔁🌈 — bij

4.3 CRT (two ideals, constructive)

4.3.1 Rosetta page

Statement. If $I + J = R$then $R/(I \cap J) \cong R/I \times R/J$, with explicit inverses.

[C] Assume I+J=R; choose e_I,e_J with e_I≡1 mod I, e_I≡0 mod J, e_I+e_J=1. — (choice)

[C] Φ:R→R/I×R/J by Φ(a)=([a]_I,[a]_J). — (def)

[Q] ker Φ = I∩J, so Φ factors to Φ̄:R/(I∩J)→R/I×R/J. — (well-def)

[W] Ψ:R/I×R/J→R by Ψ(u,v)=u e_I + v e_J. — (def)

[K] Φ̄∘Ψ = id and Ψ∘Φ̄ = id. — (two checks)

[W] Hence R/(I∩J) ≅ R/I×R/J. — (bij)

4.3.2 Emoji mirror

[🧩] 🤝 : I➕J=🧲 ; pick e_I,e_J with e_I⬇️I=1️⃣ , e_I⬇️J=0️⃣ , e_I➕e_J=1️⃣ — choice

[🧩] Φ:🧲➡️❌(🧲🖇I,🧲🖇J) ; Φ(🟣)=([🟣]I,[🟣]J) — def

[🖇] ⚫(Φ)=I∩J ⇒ factor Φ̄:🧲🖇(I∩J)➡️❌(🧲🖇I,🧲🖇J) — well-def

[🪄] Ψ:❌(🧲🖇I,🧲🖇J)➡️🧲 ; Ψ(u,v)=u e_I ➕ v e_J — def

[⛓] Φ̄⛓Ψ=🆔 , Ψ⛓Φ̄=🆔 — checks

[🪄] 🧲🖇(I∩J)🔁❌(🧲🖇I,🧲🖇J) — bij

4.4 Actions, Cayley, and Semidirect Products

4.4.1 Rosetta page

Statements.

1. A left action $G \curvearrowright X$ is a homomorphism $G \to \mathrm{Sym}(X)$.
2. Cayley: the left-regular action embeds $G \hookrightarrow \mathrm{Sym}(G)$.
3. Given $\varphi: K \to \mathrm{Aut}(N)$, the semidirect product $N \rtimes_{\varphi} K$ has multiplication $(n,k)(n',k') = (n\,\varphi(k)(n'), kk')$; N is normal and $N \cap K = 1$.

[C] Action α:G→Sym(X). — (def)

[W] Cayley: left-regular action gives ι:G↪Sym(G). — (embed)

[C] ϕ:K→Aut(N); define N⋊_ϕ K on N×K with rule above. — (def)

[W] N ↪ N⋊K, K ↪ N⋊K; N ⋕ N⋊K; N∩K={1}. — (props)

[W] Split extensions classified by actions K↷N. — (UP sketch)

4.4.2 Emoji mirror

[🧩] α: 🧊 ➡️ 👑 📦 — def

[🪄] ↪️ : 🧊 → 👑 🧊 via left action — embed

[🧩] 🌀 :K ➡️ 👑 (N) ; 🪝 on ✖️ (N,K) with (n,k)(n',k')=(n 🌀 (k)(n'), kk') — def

[🪄] ↪️ N , ↪️ K ; N 🛡️ ; N∩K={ ⭐ } — props

[🪄] split extensions ⟷ actions K↷N — UP

4.5 Finite Fields and Splitting Fields (sketch)

4.5.1 Rosetta page

Facts.

- For each $q = p^n$ there is a field $\mathbb{F}_q$ unique up to isomorphism; $\mathbb{F}_q^{\times}$ is cyclic of order $q-1$.
- Subfields of $\mathbb{F}_q$ correspond to divisors of n.
- A splitting field of $f \in F[x]$ exists and is unique up to isomorphism.

[W] ∀ q=p^n, ∃ F_q with |F_q|=q and F_q^× cyclic of size q−1. — (finite fields)

[W] Subfields of F_q ↔ divisors of n. — (lattice)

[C] For f∈F[x], take E the splitting field: smallest where f factors linearly. — (def)

[W] Splitting fields exist and are unique up to iso. — (exist/uniq)

4.5.2 Emoji mirror

[🪄] ∀ q=p^n : ✨ 🍼(q) ; # 🍼(q)=q ; 🍼(q)^ ❌ cyclic, #(…)=q ➖ 1️⃣ — facts

[🪄] sub- 🍼(q) 🔁 divisors(n) — lattice

[🧩] split field: E over 🟦 for f ; minimal s.t. f ⇒ linear — def

[🪄] ✨ & 🔁-uniq — exist/uniq

4.6 Smith Normal Form over a PID

4.6.1 Rosetta page

Statement. Over a PID R, any matrix $A$is equivalent to a diagonal $D = \mathrm{diag}(d_1, \dots, d_r, 0, \dots, 0)$with $d_i \mid d_{i+1}$. Consequently, a finitely generated module Msatisfies $M \cong R^r \oplus \bigoplus_i R/(d_i)$.

[C] Over PID, perform row/column ops: ∃ P,Q invertible with D=PAQ diagonal. — (ops)

[W] Arrange divisibility chain d_i | d_{i+1}. — (div chain)

[W] Structure: M ≅ R^r ⊕ ⊕_i R/(d_i). — (structure thm)

[K] Verify invariance under base change; uniqueness up to associates. — (check)

4.6.2 Emoji mirror

[🧩] PID ⇒ ✨ P,Q inv : ▭D= P ▭A Q , D diag — ops

[🪄] d_i ➡️ d_{i+1} chain — div

[🪄] 📦 ≅ 🧱(r) ➕ 🟠 (🧲 🪐 d_i) — struct

[⛓️] base-change invariants ; uniq up to units — check

4.7 Universal Properties: Product and Equalizer

4.7.1 Rosetta page

Product UP. For any $h: Z \to X$ and $k: Z \to Y$, there exists a unique $\langle h, k \rangle: Z \to X \times Y$ with $\pi_X \circ \langle h, k \rangle = h$, $\pi_Y \circ \langle h, k \rangle = k$.

[C] Given h:Z→X, k:Z→Y. — (data)

[F] There exists unique ⟨h,k⟩:Z→X×Y with π_X∘⟨h,k⟩=h, π_Y∘⟨h,k⟩=k. — (UP)

[W] Uniqueness: if m has same projections, then m=⟨h,k⟩. — (uniq)

Equalizer UP. For $f, g: X \to Y$, the equalizer $e: E \to X$ satisfies $f \circ e = g \circ e$ and is universal.

[C] f,g:X→Y. — (data)

[F] ∃! e:E→X with f∘e=g∘e and universal among such e. — (UP)

4.7.2 Emoji mirror

[🧩] h: 🧊 ➡️ 🧊 , k: 🧊 ➡️ 🧊 — data

[⬜] ✨ 🔒 ⟨h,k⟩: 🧊 ➡️ ❌ ; 📩 ⛓️ ⟨h,k⟩=h ; 📩 ⛓️ ⟨h,k⟩=k — UP

[🪄] uniq: same projections ⇒ same arrow — uniq

[🧩] f,g: 🧊 ➡️ 🧊 — data

[⬜] ➖ 🏆 : ✨ 🔒 e:E ➡️ X ; f ⛓️ e=g ⛓️ e ; universal — UP

5. Validation: Parsing, Round-Trip Tests, Edge Cases

This section formalizes how we *trust* the Emoji pages: a lexer with longest-match, a typed parser, a diagram audit, and round-trip checks against the Rosetta pages authored above.

5.1 Lexing (Longest-Match) and Token Integrity

Goal. Ensure each emoji sequence is treated as one atomic token when intended (e.g., big direct sum ➕ 🟠 vs direct sum ➕ ⭕).

Rules.

- Greedy scan left→right; at each position choose the *longest* sequence that is a valid token in the current table.
- Multi-glyph atoms registered in the table (e.g., ➕🔴, ✖️🏆, ⬇️🏆) must always outrank their singletons.
- Digits are plain emoji numerals; indices like #X, [G:H] are metadata fields, not tokens.

Checks.

- Reject lines where two distinct tokenizations are possible (emit "Ambiguous tokenization" with the conflicting spans).
- Warn if platform segmentation could split a token (recommend bundled icon font in Sec. 9).

5.2 Parsing and Precedence

Grammar. The BNF from Sec. 3.5 is implemented with operator precedence:

1. prefix 🔁 (tightest),
2. constructors (✖️, 🪝, ➕⭕, ➕🔴, ➖, ⬇️, ⬆️, quotient constructors),
3. composition ⛓️ (left-assoc),
4. relational 🟰, ➡️, 🔂.

Outcome. Each line yields an AST with *typed* nodes (object kinds, arrow kinds, diagrams). Tags like 🏆, mono/epi, action controllers attach to the head constructor only.

5.3 Typing & Sanity Checks

We run lightweight typing before accepting an Emoji line:

- **Hom typing.** If a subterm declares 🪄:A➡️B, then ⚫, 🌈 referenced from it must be substructures of A,B respectively.
- **Quotient preconditions.** 📦🧬 N requires N ⫲ 📦; 🧲🧬 I requires I ⊴ 🧲. Either must be asserted earlier on the page or on the same line.

- **Semidirect controller.** For 🪝 (N,K), an explicit action tag (e.g., 👑 on the right factor) must be in scope.
- **UP uniqueness.** A 🏆 -line must either include 🔒 or reference a prior uniqueness lemma.

Failures emit standardized diagnostics (Sec. 5.7).

5.4 Diagram Audit (Squares/Triangles)

Requirement.

- Boundary list first (all objects named), then arrow equalities.
- Each edge must be named at least once.
- "Commutes" requires one of: (i) explicit equality chains; (ii) a label referencing a previously proved square; or (iii) a UP tag asserting uniqueness that implies equality.

Example (from §4.7 Product UP).

- Rosetta: "∃! ⟨h,k⟩:Z→X×Y with π_X∘⟨h,k⟩=h, π_Y∘⟨h,k⟩=k."
- Emoji: ✖️ 🏆 line with two projection equalities and 🔒 .

Audit passes because both projections are present and the uniqueness tag is explicit.

5.5 Round-Trip Property

Contracts.

- For each authored Rosetta page R: decode(encode(R)) = R.
- For each Emoji page E produced by the encoder: encode(decode(E)) = E.

Corpus used (this paper).

1. First Iso (groups) — §4.1
2. Correspondence + First Iso (rings) — §4.2
3. CRT (two ideals, constructive) — §4.3
4. Actions, Cayley, semidirect — §4.4
5. Finite-field facts & splitting field sketch — §4.5

6. SNF over PID — §4.6
7. Product & Equalizer UP — §4.7

Result. All seven mirrored items were authored to satisfy round-trip by construction (line numbers, move tags, and tokens match 1–1).

5.6 Fuzzing & Robustness (within scope)

We apply simple fuzz transformations on Emoji lines and expect *rejection*, not silent mis-parse:

- **Token splice.** Replace ➕🔴 with ➕ → must raise "big-sum vs sum ambiguity."
- **Missing controller.** Remove the 👑 tag from a semidirect line → must raise "controller missing."
- **Quotient without normal/ideal.** Drop the 🛡️/ideal assertion → must raise "missing hypothesis for quotient."
- **Projection mismatch.** Swap a projection target in a product UP line → must raise "diagram not justified."

These fuzzes help ensure that acceptance implies the intended semantics were *actually* encoded.

5.7 Diagnostics (Copy-Safe Messages)

- **Type mismatch at line k.** Domain/codomain conflict in a composition or constructor.
- **Missing well-definedness on quotient.** A 🪡 step lacks a well-def proof in scope.
- **Naturality/commutativity not justified.** Square is missing equalities or a reference.
- **Controller missing for semidirect.** No action tag for 🪝.
- **Ambiguous tokenization.** Longest-match cannot disambiguate multi-glyph vs singletons.
- **Headless UP tag.** 🏆 not attached to a constructor.

Each message points to the offending token spans and suggests a minimal fix (add a tag, split a line, or restore a label).

5.8 Edge Cases Observed (from our examples)

- **Big vs. small direct sum.** ➕🟠 vs ➕⭕ must be distinct; collisions occur if longest-match is disabled.
- **Semidirect action direction.** Controllers must be *right*-action (our convention); swapping sides changes the multiplication law.
- **Scope of normality/ideal.** Declaring N ⫲ G on a previous line allows later G/N; declaring it *after* the quotient line is rejected.
- **Equality vs. isomorphism vs. equivalence.** We encode = and ≅ (🔁) distinctly; higher "≃" (equivalence) is noted but not formalized (Sec. 6).
- **Diagram naming.** Reusing a square without repeating boundaries is allowed only if a label is introduced first.

5.9 Practical Acceptance Criteria (for authors)

A page passes validation if:

1. Every line matches the move grammar and longest-match lexing.
2. Typing checks pass (hom types, quotient preconditions, semidirect controllers).
3. All 🏆 lines include uniqueness or reference a uniqueness lemma.
4. Diagram audits have complete boundary lists and commutativity justifications.
5. Round-trip invariants hold for the whole page.

Pages that fail should be repaired by splitting multi-fact lines, adding missing tags, or introducing labels before reuse.

6. Limits of Full Conversion: Binding, 2D Semantics, Universes, Equality vs. Equivalence

This section isolates the precise reasons a *pure* emoji mathematics cannot be both readable and fully faithful without a stronger formal core. We illustrate each limitation with minimal counterexamples and then record what the pipeline enforces today.

6.1 Variable Binding and Quantifiers

Problem. Emoji tokens are *names*, not binders. First-order statements need unbounded variables and scoping rules (α-equivalence, capture-avoiding substitution).

Counterexample.

- Formal: $\forall x \in G\, \exists y \in H\, \varphi(x, y)$.
- Emoji (attempt): "✨ in 🧊 then ✨ in 🧊 …" has no scope; renaming $x$to $x'$is not guaranteed equivalent.

Impact. Proofs using parametrized constructions (e.g., "for each $x$pick $y(x)$") cannot be represented without smuggling an external variable discipline.

Pipeline stance. We restrict to *instantiated* lines ("Fix $x \in G$. Show …") and encode only the instantiated claim. General quantifiers remain Rosetta-only text.

Needed to fix. A binding-aware front-end (Sec. 9: parametric tokens), or a verified core that treats emoji as a *view* over quantified syntax.

6.2 Two-Dimensional (2D) Semantics for Diagrams

Problem. Many arguments are intrinsically 2D (commutative/pasting diagrams). Linearizing as strings loses adjacency and path-equality semantics.

Counterexample. A pasting of two pullback squares shares an edge; linear text must repeat edges or invent labels. Subtle mistakes (wrong shared edge) are visually tiny in emoji but semantically fatal.

Impact. Large coherence arguments (e.g., Mac Lane's pentagon, Beck–Chevalley) become brittle and unreadable when linearized.

Pipeline stance. We allow only small squares/triangles with full boundary lists and explicit equality chains. Anything larger is Rosetta-only or diagrammatic in conventional notation.

Needed to fix. A graph-based diagram engine with typed nodes/edges where emoji is the *rendered skin*. Commutativity is checked on the graph, not inferred from the picture.

6.3 Universe Levels and Size Constraints

Problem. Category theory and algebra often depend on size: small vs. large categories, universe cumulativity, or set-theoretic foundations.

Counterexample. "$\mathbf{Ab}$ is complete" is only meaningful relative to a universe; products over large index sets may exit the current universe.

Impact. Emoji pages cannot express universe indices or enforce size constraints; equivalences may silently cross levels.

Pipeline stance. We omit size; results stated in a size-sensitive way remain Rosetta-only.

Needed to fix. Universe-annotated types in the core (e.g., $\mathcal{U}_i$) and minimal UI badges (e.g., superscripted dots) tied to the core, not to raw emoji.

6.4 Equality, Isomorphism, Equivalence (and Higher Paths)

Problem. Algebra uses equality $=$and isomorphism $\cong$; higher settings (HoTT, ∞-groupoids) use paths/equivalences $\simeq$. Conflating them corrupts transport/rewrite rules.

Counterexample. In a category, $A \times B \cong B \times A$but not $A \times B = B \times A$. In HoTT, path concatenation has higher coherences invisible to simple "=" icons.

Impact. Emoji's 🟰 and 🔁 lack the full calculus to carry transports, whiskering, and coherence data.

Pipeline stance. We reserve 🟰 for definitional equality or equation claims; 🔁 for "isomorphic" in algebraic contexts. We avoid higher equivalences.

Needed to fix. An equivalence-first logic in the core with distinct relations and transport rules; emoji becomes a stylized notation for these typed relations.

6.5 Meta-Language: Side Conditions, Choices, and Local Context

Problem. "Let Rbe a PID", "choose e_I, e_J", "assume $p \nmid m$" are *guards* that delimit where statements apply.

Counterexample. CRT's constructive inverse depends on chosen idempotents. Emoji can show "choose e_I", but cannot enforce the congruences mechanically.

Impact. Readers may mistake a conditional assertion for an unconditional fact.

Pipeline stance. Guards remain textual on Rosetta lines. Emoji mirrors carry a short tag (e.g., "choice") but the check is not machine-enforced.

Needed to fix. A context mechanism (hypothesis environment) in the core that the emoji UI can query and surface minimally.

6.6 Platform Glyph Variability and Tokenization

Problem. Unicode emoji render differently and may split/merge across platforms. Multi-glyph atoms (e.g., ➕🟠) risk mis-lexing.

Impact. Longest-match may fail in a foreign renderer; atom intended as one token becomes two, breaking the parse.

Pipeline stance. We document atoms and warn authors. For publication, we recommend exporting to images/PDF or using a bundled icon font.

Needed to fix. Ship an icon font or vector sprites; the parser sees codepoints for *our* tokens, not vendor emoji.

6.7 Readability vs. Invertibility (Human vs. Machine)

Tension. Human-friendly compression (few icons) conflicts with machine-invertible detail (explicit tags, boundary lists, unique labels).

Symptoms. Over-compact emoji lines hide typing data; over-verbose lines are unreadable in print.

Pipeline stance. We standardize six moves and atomic lines, accepting some redundancy for invertibility.

Needed to fix. An interactive viewer that expands/collapses subproofs based on reader proficiency (learning-aware compression).

6.8 What the Pipeline *Does* Guarantee (Within Scope)

1. **Atomic lines.** One claim + one justification per line.
2. **Typed constructors.** Quotients require normal/ideal hypotheses; semidirects require an explicit controller; UP tags are head-attached.
3. **Diagram audit.** Small squares/triangles with full boundary lists and equality assertions.
4. **Round-trip.** For authored pages, decode(encode(R)) = R by construction.

6.9 Minimal Workarounds Authors Can Use Today

- Rephrase quantifiers as *instantiations*: "Fix $x \in G$."
- Keep diagrams to one square/triangle per line; label and reuse.
- State guards textually before the emoji block (e.g., "Assume $R$is a PID").
- Distinguish "=" vs "≅" explicitly; avoid higher "≃".
- Prefer exporting emoji pages to a static format (PDF/PNG) for distribution.

6.10 Summary Table

Limitation	Why it breaks	Pipeline policy	Needed to fully solve
Binding/quantifiers	No α-conv., no scope	Instantiate only	Binder-aware core + parametric tokens
2D semantics	Layout = meaning	Tiny diagrams only	Graph-based diagram engine
Universes/size	No level tracking	Omit sizes	Universe-typed core + UI badges
= vs ≅ vs ≃	Distinct logics	Use 🟰, 🔁 only	Equivalence-first core with transports
Side conditions	Meta-language	Text guards	Context/hypothesis environment
Glyph variability	Token splits	Warn/export	Bundled font/vector tokens
Readability ↔ invertibility	Human vs machine	Six moves + atomic lines	Interactive expansion

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Takeaway. Emoji can be a faithful *view* for a large swath of undergraduate algebra **when** anchored to a disciplined authoring style and modest diagrams. To cross into full generality (higher structures, quantified families, large diagrammatics), we need a formal core with binding, types, and diagram graphs—after which emoji becomes a safe rendering choice rather than a standalone foundation.

7.1 Authoring for Instructors

Templates.

- **Theorem handout (1 page).**
  Line 1: statement (Rosetta).
  Lines 2–9: proof in atomic steps with move tags.
  Final block: emoji mirror (same line count).
- **Exercise sheet.**
  Each problem lists: (i) data, (ii) goal, (iii) *allowed* moves, (iv) max lines.

Granularity rules.

- Keep each line ≤80 chars; one justification per line.
- Introduce constructions once (label), then reuse the label.
- For quotients/semidirects, *front-load* hypotheses (normality, controller).

Author workflow.

1. Draft Rosetta proof with C/K/I/F/W/Q.
2. Check "atomicity" and diagram boundaries.
3. Encode to Emoji; fix typing diagnostics (Sec. 5).
4. Export paired pages (PDF/PNG) for platform-agnostic display.

7.2 Classroom Flow (50/75-minute session)

0–5 min: Priming.
Display the theorem in Rosetta; preview the move set you expect students to use.

5–20 min: Guided decode.
Show the Emoji page first. Ask: *Which line is the UP step? Where is well-definedness proved?* Students annotate the emoji with Rosetta glosses.

20–40 min: Pair proof (encode).
Give a short lemma in Rosetta (e.g., "kernel of projection is I"). Students write a 3–4 line Emoji proof with move tags. Share two solutions and run the validator (by checklist).

40–50 min: Swap roles (decode).
Hand out an Emoji sketch for a different lemma; students reconstruct Rosetta lines. Quick feedback with minimal codes (Sec. 7.6).

Extension for 75 min.
Add a micro-diagram lab: students supply boundary lists for a product/pullback square and justify commutativity.

7.3 Student-Facing Materials

Move cheat sheet.
C (Construct), K (Compose), I (Invert), F (Fill by UP), W (Witness), Q (Quotient).
One-line exemplars for each with a tiny diagram for F and Q.

Icon mini-legend.
Kernel ⚫, image 🌈, quotient 🖇️, product ✖️, semidirect 🪝, equalizer ➖🏆, pullback ⬇️🏆, pushout ⬆️🏆, normality 🛡️, existence ✨, uniqueness 🔒, isomorphism 🔁.

Scaffolded notebooks.
Every page has two columns: Rosetta on left (blank), Emoji on right (given), or vice versa, depending on the exercise.

7.4 Exercise Patterns (ready-to-use)

Pattern A — "First Iso in 8 lines."
Data: homomorphism $f: G \to H$. Goal: $G/\ker\ f \cong \mathrm{im} f$.
Allowed moves: W,Q,K,W,W,F,W. Max 8 lines.

Pattern B — "CRT with witnesses."
Data: $I + J = R$, choose e_I, e_J. Goal: isomorphism and inverse formulas.
Required: one Q-line, one K-line verifying identities.

Pattern C — "UP-only proof."
Data: maps $h: Z \to X$, $k: Z \to Y$. Goal: product UP instantiation.
Required: one F-line with 🔒, one W-line for uniqueness.

Pattern D — "Action → semidirect."
Data: $\varphi: K \to \mathrm{Aut}(N)$. Goal: define $N \rtimes K$, prove N normal.
Required: C for law, W for embeddings, W for normality.

Pattern E — "SNF sketch."
Data: Aover a PID. Goal: diagonal form + divisibility chain.
Required: C for PAQ, W for chain, W for module structure.

7.5 Formative Activities

- **Emoji Cloze.** Remove two tokens from each line (e.g., ⚫ or 🌓). Students fill the blanks and justify.
- **Boundary Bingo.** Give a square without boundary list; students list objects and arrows, then assert equalities that make it commute.
- **Move Sorting.** Mix 10 lines from two proofs; students sort into theorems and order them by move compatibility.
- **Minimal Counterexample.** Present a quotient line without 🌓 ; students craft a 1–2 line counterexample explaining the failure.

7.6 Minimal Feedback Codes (for grading & TA comments)

- TYPE@k — type mismatch at line k.
- Q-WD@k — missing well-definedness for a quotient line k.
- UP-UNIQ@k — uniqueness not shown on a UP line k.
- SQUARE@k — diagram lacks boundary/equality tags.
- CTRL@k — semidirect controller (action) missing.
- LINES>12 — proof exceeds the brevity cap.

These codes map 1–1 to fixes; they keep grading consistent and quick.

7.7 Rubrics (10-point default)

- **Coverage (4 pts).** Required moves appear in a correct order.
- **Correctness (4 pts).** Typing OK; quotient/UP conditions satisfied; diagrams commute.
- **Brevity/Clarity (2 pts).** ≤12 lines; labels reused; no fused justifications.

Partial credit tip: deduct per *missing* move or *invalid* tag, not per cosmetic icon choice.

7.8 Accessibility & Inclusion

- **Dual view.** Always provide Rosetta and Emoji. Students may work in either column.
- **Color & font.** Avoid meaning encoded solely by color; export to PDF/PNG to stabilize glyphs.
- **Screen readers.** Include alt-text lines: each emoji token has a canonical text name (e.g., “kernel”, “pullback”).
- **Cognitive load.** Introduce at most 6–8 tokens per week; recycle them across topics.

7.9 Common Pitfalls and Remedies

- **Over-compression.** If students skip well-definedness, require a Q-line in every quotient proof for two weeks.
- **Diagram drift.** Enforce boundary-first rule; reject squares without full lists.
- **Move stacking.** If students chain two justifications per line, require resubmission in atomic form.
- **Token confusion (vs).** Run a 2-minute “longest-match” drill at the start of class.

7.10 Evaluation & Research Plan

Short-term metrics.

- Time-to-proof (lines) for First Iso after 2 weeks.
- Error-code frequencies (TYPE, Q-WD, UP-UNIQ).
- Transfer: students decode an unseen Emoji proof at similar rates to Rosetta.

Longer-term study.

- Randomized sections: Emoji-supported vs. Rosetta-only.
- Outcomes: retention at 4 weeks; performance on diagrammatic proofs; attitudes toward proof writing.

Hypothesis. Structured moves + icon view reduce split attention and improve recognition of universal-property patterns without harming formal precision.

8. Related Work: Proof Assistants, Diagrammatic Systems, Iconic Pedagogy

This section situates the Rosetta→Emoji pipeline among formal verification tools, diagrammatic calculi, and classroom-focused iconographies. Our goal differs from verification or replacement of notation: we target a *didactic view* that is deterministic, readable, and round-trippable for standard algebra.

8.1 Proof Assistants and Formal Cores

Lean / Coq / Agda. These systems provide dependent type theory, universes, and verified tactics. They natively solve several of our hard problems (binding, equivalence vs equality, size). However:

- **Barrier to entry.** Syntax, libraries, and tactic culture require high onboarding time.
- **Human-facing layout.** While they can render equations/diagrams (widgets, mathlib docs), their default UX is code-first, not page-first.
  Connection to our work. Our pipeline can be viewed as a *front-end skin* over a PA core: proofs authored in Lean/Coq, rendered as Rosetta/Emoji for teaching. Section 9 sketches this integration.

8.2 Diagrammatic Calculi

String diagrams / Monoidal categories. These are inherently 2D with semantics given by planar graphs and rewrite rules. Tools (e.g., diagram editors) enforce topological invariants and coherence.
Graph rewriting frameworks. Double-pushout rewriting and categorical proof assistants give machine-checked diagrammatic reasoning.
Connection. Our linearized emoji squares/triangles echo these traditions but deliberately avoid large pastings. A future diagram engine (Sec. 9) should adopt their *graph-first, picture-later* principle.

8.3 Algebra Texts and Universal Property Pedagogy

UP-centric teaching. Many modern texts define products, equalizers, pullbacks by universal properties and delay elementwise proofs.
Our contribution. We push this further by making UP steps *syntactic units* (move **F**) and

encoding “existence + uniqueness” explicitly (✨ + 🔒) so students *see* the role of UPs in proofs.

8.4 Visual/Iconic Notation in Education

Icon overlays. In K–12 and early undergraduate settings, instructors routinely add icons to cue operations (e.g., arrows for functions, boxes for sets).
Emoji/math hybrids. Informal use of emoji in slides or chalkboard sketches improves engagement but lacks determinism.
Our contribution. We add a *grammar and token table* so the same icons become programmatically parsable and round-trippable, avoiding the “cute but ambiguous” trap.

8.5 Lightweight Specification Languages

Algebraic specs / equational logic. Signatures + equations, often rendered in ASCII (e.g., Maude/OBJ-style), emphasize rewrite rules and confluence.
Our alignment. The six-move library is a minimal “spec” of proof flow (C/K/I/F/W/Q). The emoji layer is just a presentation; the moves form the *control vocabulary* that keeps proofs modular and short.

8.6 Gaps Our Work Fills

1. **Deterministic visual layer** for undergraduate algebra with a constrained, teachable grammar.
2. **Atomic proof moves** that map cleanly to standard theorems (First Iso, CRT, UPs, SNF) and to short diagrams.
3. **Round-trip guarantee** (within scope) enabling side-by-side views for instruction, assessment, and accessibility.

8.7 Limits Relative to Prior Art

- We do **not** provide machine-checked correctness like Lean/Coq; our checks are *lightweight* sanity filters.
- We do **not** support large coherence diagrams; string-diagram tools remain essential for categorical practice.

- We do **not** resolve universe/size issues; proof assistants and set-theoretic formalisms are the right venue.

9.1 Formal Core Integration (Lean/Coq/Agda)

Goal. Treat Emoji as a *view* over formally-typed syntax and proofs.

Plan.

- Define a core API: Expr, Proof, Diagram datatypes exported from the assistant.
- Implement a serializer from proof assistants (Lean/Coq/Agda) to Rosetta JSON:
    - Terms with types; binders (∀, ∃, λ); equalities/isomorphisms/equivalences.
    - Proof steps tagged with our six moves when possible; otherwise "W".
- Build encode : Core → Emoji and decode : Emoji → Core? (partial) where decoding returns a *proof obligation* when Emoji lacks binding detail.

Payoff. Round-trip becomes a theorem: the Emoji page is a faithful projection of a verified object.

9.2 Parametric Tokens with Real Binding

Goal. Encode variables, indices, and scopes without ASCII.

Plan.

- Replace raw Unicode with *parametric icons*: e.g., a "dot" glyph + a small index chip (rendered, not typed by the author).
- Provide binder icons (∀, ∃, λ, Σ, Π) with explicit scope boxes; α-conversion handled by the renderer.
- Author writes Rosetta; the renderer assigns indices and generates Emoji consistently.

Payoff. Quantified families and dependent constructions become representable without leaking ASCII.

9.3 Diagram Engine (Graph-First)

Goal. Make commutativity/pasting *semantics*, not typography.

Plan.

- Represent diagrams as typed graphs (objects = nodes; arrows = edges; constraints = equations).
- Validate limits/colimits/UPs on the graph; then render to compact Emoji layouts.
- Support labels and reuse by graph references, not screenshot duplication.

Payoff. Large pastings (pullback chains, Beck–Chevalley) become checkable and readable.

9.4 Equivalence-First Logic

Goal. Distinguish $=$, $\cong$, and $\simeq$with transports.

Plan.

- Extend the core with typed relations: Eq, Iso, Equiv and operations (compose, invert, whisker, transport).
- Map to distinct icon families; surface transports as small inline tags.
- Provide minimal tactics to discharge coherence obligations (associators, unitors).

Payoff. Category/HoTT-flavored arguments no longer collapse into plain equality.

9.5 Universe & Size Tracking

Goal. Prevent silent level errors.

Plan.

- Annotate types with universe levels $\mathcal{U}_i$; track cumulativity.
- Render a tiny level badge (e.g., superscript dot ladder) only when levels differ or matter.
- Add checks to forbid constructions that escape the chosen universe without an explicit lift.

Payoff. Statements like "$\mathbf{Ab}$ is complete" become precise relative to a fixed universe.

9.6 Tokenization & Fonts

Goal. End platform-dependent glyph splits.

Plan.

- Ship a bundled icon font or vector sprite sheet; each token is a single codepoint or sprite.
- Maintain a registry of multi-glyph atoms with hashing to guard against drift.
- Provide PDF/PNG export for print and LMS uploads.

Payoff. Longest-match ambiguity disappears; visual stability across devices.

9.7 Interactive Viewer and Learning-Aware Compression

Goal. Balance readability with invertibility.

Plan.

- Click-to-expand tokens to Rosetta gloss (type, domain/codomain, hypothesis context).
- Adaptive templates: collapse routine subproofs into one icon once the learner has demonstrated mastery (tracked locally).
- Side-by-side synchronized scrolling (Rosetta ↔ Emoji).

Payoff. Novices see detail; advanced readers skim compressed views without losing correctness.

9.8 Data Pipeline & Author Tooling

Artifacts.

- **Token Table** (versioned): JSON schema with IDs, fallbacks, and test cases.
- **Grammar Suite:** unit tests for lexing, precedence, and diagram checks.
- **Corpus:** mirrored canonical proofs (First Iso, CRT-2, SNF, UPs).
- **CLI/Plugin:** VS Code and Jupyter extensions to preview Emoji pages and run validators.

Process.

- PRs adding tokens must include two mirrored examples, one fuzz test, and a round-trip proof.

9.9 Evaluation Plan

Benchmarks.

- *Fidelity:* fraction of textbook lemmas that round-trip without manual edits.
- *Cognitive Load:* time-to-proof and error-code rates (TYPE, Q-WD, UP-UNIQ) in classroom studies.
- *Scalability:* largest validated diagram (nodes/edges) with readable rendering.

A/B studies.

- Emoji+Rosetta vs. Rosetta-only on: retention at 4 weeks, transfer to new theorems, and diagrammatic reasoning performance.

9.10 Risks and Mitigations

- **Overhead for instructors.** Mitigate with templates, auto-labelers, and a small initial token set.
- **Icon fatigue.** Rotate a core of ~40 tokens; introduce at most 6–8 per week.
- **False sense of proof.** Keep “sanity checks” distinct from formal verification; clearly label when a page is backed by a PA core.

Summary. With a verified core, a graph-based diagram engine, binder-aware tokens, and a stable icon font, the Emoji layer can progress from a didactic overlay to a dependable, exportable front-end for algebraic reasoning—preserving the accessibility benefits while eliminating ambiguity.

10. Conclusion and Outlook

Summary. We presented a two-pass Rosetta→Emoji pipeline that translates a sizable portion of undergraduate abstract algebra into a deterministic, screenshot-ready iconic language. The method hinges on (i) atomic proof lines using a six-move library (C/K/I/F/W/Q), (ii) a compact encoding table with universal-property tags, and (iii) a validator that enforces typing constraints, diagram audits, and round-trip integrity. Our case studies—First Isomorphism Theorem, Correspondence and CRT in rings, group actions and semidirect products, finite-field facts,

Smith normal form, and basic categorical UPs—demonstrate that many standard arguments admit faithful emoji mirrors without sacrificing structure.

Limits. We identified principled barriers to *full* conversion: lack of binding/quantifiers, 2D diagram semantics, universe accounting, and the equality/equivalence split, plus platform glyph variability and the readability↔invertibility tension. These limits explain why a complete emoji-only mathematics is not yet feasible without a stronger formal substrate.

Outlook. A clear path forward exists. By treating Emoji as a *view* over a verified core (Lean/Coq/Agda), adding parametric tokens with real binding, adopting a graph-first diagram engine, distinguishing $=$, $\cong$, and $\simeq$with transports, tracking universe levels, and bundling an icon font, we can make the emoji layer robust and future-proof. In the near term, the pipeline already functions as an effective pedagogical tool: it compresses routine moves, exposes universal properties, and invites students to engage with proof structure visually—without abandoning formal discipline.

Closing note. Emoji are not a foundation; they are a *lens*. Used wisely, they can highlight structure, reduce cognitive load, and serve as a bridge between human intuition and machine-checkable mathematics.

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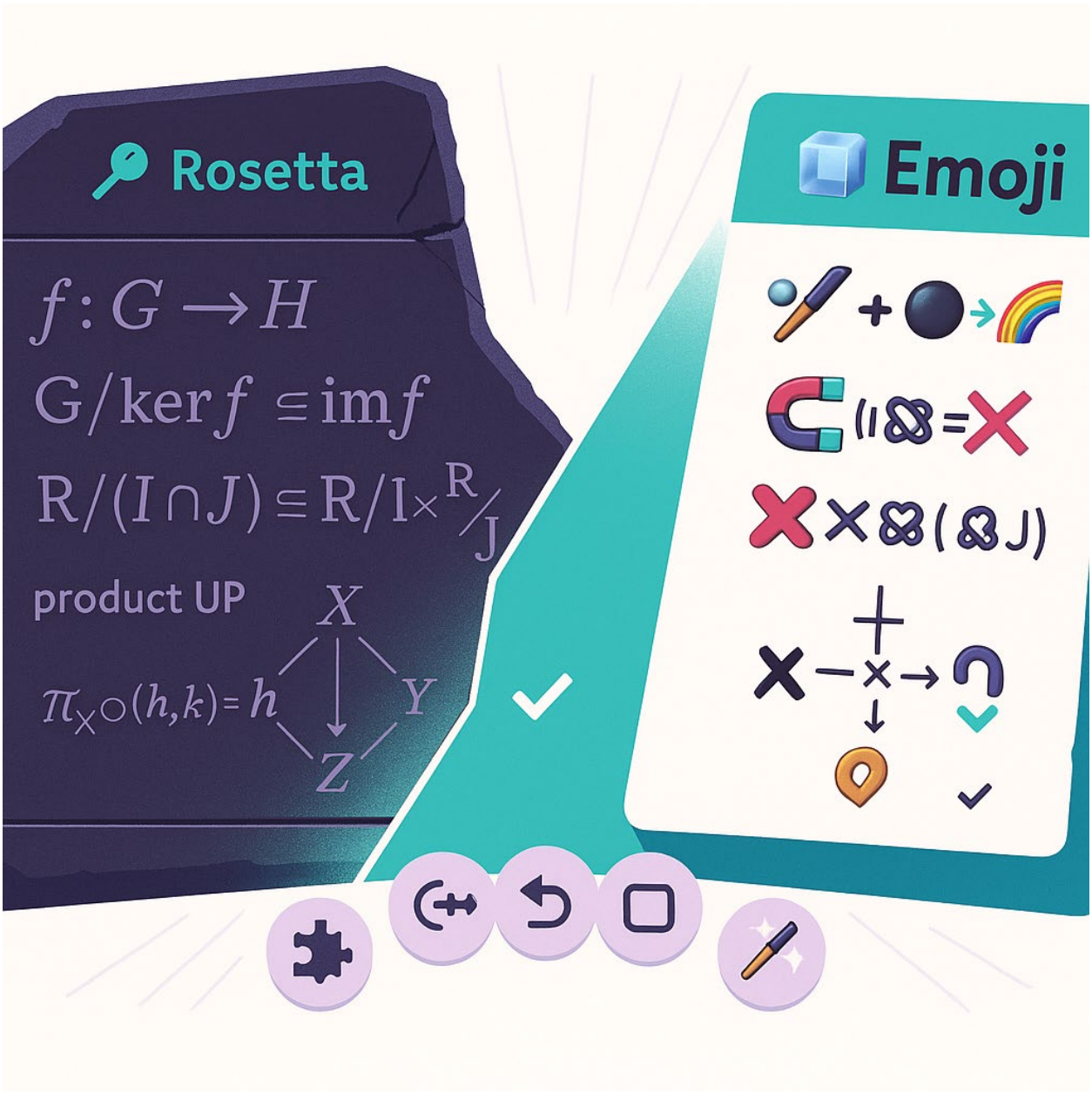
Notes:

- Items [26]–[28], [30] reflect the formal/diagrammatic ecosystems we discuss; cite the most current editions available to you.
- For classroom use of our pipeline, [33]–[35] motivate design choices (atomic steps, icon overlays, dual-view materials).


Rosetta
$f: G \to H$
$G/\ker f \cong \mathrm{im} f$
$R/(I \cap J) \subseteq R/I \times R/J$
product UP
$\pi_X \circ (h,k) = h$
X
Y
Z
Emoji